The pitching motion of a circular disk

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The interaction of a pitching circular disk with the motion induced by the disk in the surrounding fluid is investigated in this paper. MacCamy's (1961) method of simplifying the three-dimensional problem of a circular disk to the two-dimensional problem is found to apply in the present analysis. The integral equation is solved numerically to determine the dependence of pressure, added moment of inertia, and damping coefficient on the frequency of the oscillation.

1. Introduction

This paper is concerned with the forced pitching motion of a circular disk on an inviscid, incompressible fluid. The forced motion is assumed to be simple harmonic in time. If the pitching amplitude is small, the amplitude of the resulting waves will also be small in comparison with their length. Therefore the problem can be linearized by neglecting higher-order terms in the boundary conditions.

Several authors have considered the problem of forced water waves with special cross-section geometry, and a summary of their papers was made by Wehausen & Laitone (1960). A method of solving the problem of an oscillating obstacle on a free surface by means of surface distributed sources has been given by John (1950). Properties of the distributed sources lead one to formulate the problem with an integral equation for determining the unknown density. For the case of a disk, the kernel of the integral equation does not involve the spatial derivatives, and moreover it can be evaluated explicitly.

For the problem of the circular disk the known boundary value on the immersed surface is a simple product of the radial and angular variables. Therefore, following MacCamy's method, the integral equation can be simplified to contain only the radial variable.

2. General formulation

Suppose the region $\bar{y} < 0$ is occupied by an inviscid, incompressible fluid. Consider a rigid circular disk of radius $\bar{a}$ placed on the undisturbed free surface, $\bar{y} = 0$, with its centre at the origin of rectangular co-ordinates $\bar{x}$ and $\bar{z}$. The fluid motion is created by causing the disk to pitch about the $\bar{z}$-axis. It is assumed that the forced motion is periodic in time with frequency σ and is of small enough amplitude so that small-wave theory can be applied to the resulting motion.

If we call the angle which the surface of the disk makes with the $\bar{x}$-axis $\Theta(t)$, the time-periodic pitch motion can be expressed as

$$\Theta(t) = \mathrm{Re}[\Theta_0 e^{-i\sigma t}], \tag{2.1}$$

where Θ_0 is the amplitude of the forced motion. After sufficient time has elapsed for the transients to disappear, the fluid motion becomes time-periodic with frequency σ. Now, assuming irrotational motion, we introduce the time-periodic velocity potential

$$\Phi(\bar{x}, \bar{y}, \bar{z};\, t) = \mathrm{Re}[V(\bar{x}, \bar{y}, \bar{z})\, e^{-i\sigma t}]. \tag{2.2}$$

Incompressibility then implies that

$$\nabla^2 V(\bar{x}, \bar{y}, \bar{z}) = 0 \quad \text{in} \quad \bar{y} < 0. \tag{2.3}$$

Pitching of small amplitude will generate waves with amplitudes small in comparison with their wavelengths. For a free surface of small slope, the pressure and the surface elevation are given by

$$P(\bar{x}, \bar{y}, \bar{z};\, t) = -\rho^* g\bar{y} - \rho^* \Phi_t(\bar{x}, \bar{y}, \bar{z};\, t), \tag{2.4}$$

and

$$\bar{\eta}(\bar{x}, \bar{z};\, t) = -g^{-1}\Phi_t(\bar{x}, 0, \bar{z};\, t), \tag{2.5}$$

where ρ^* is the density of the fluids, g being the acceleration of gravity. Hence, the linearized free surface condition becomes

$$\Phi_{tt} + g\Phi_{\bar{y}} = 0 \tag{2.6}$$

or

$$V_{\bar{y}} - kV = 0 \quad \text{on} \quad \bar{y} = 0 \tag{2.7}$$

outside the disk, where k represents the wave-number which is equal to $\sigma^2/g = 2\pi/\lambda$, λ being the length of free waves.

On the surface of the disk, the kinematic condition to be satisfied is

$$\Phi_{\bar{y}} = \bar{x}\dot{\Theta}. \tag{2.8}$$

We note here that the immersed surface in motion differs from that in the undisturbed position. However, as a consequence of the linearization, the condition (2.8) is to be satisfied on the latter surface, then

$$V_{\bar{y}} = -i\sigma\Theta_0 \bar{x} \quad \text{on the disk.} \tag{2.9}$$

Finally, at a large distance from the disk, the propagating disturbance should be a radially outgoing regular wave, i.e.

$$V(\bar{r}, \theta, \bar{y}) - f(\theta)\,\bar{r}^{-\frac{1}{2}} \exp[k\bar{y} + ik\bar{r}] = O(1/\bar{r}) \quad \text{as} \quad r \to \infty, \tag{2.10}$$

where $\bar{r}^2 = \bar{x}^2 + \bar{z}^2$, and $\theta = \tan^{-1}(\bar{z}/\bar{x})$.

We transform the space variables and the other relevant variable into the dimensionless forms

$$x = \bar{x}/\bar{a}, \quad y = \bar{y}/\bar{a}, \quad z = \bar{z}/\bar{a}, \quad a = k\bar{a}, \tag{2.11}$$

where a is the frequency parameter which is equal to $\sigma^2\bar{a}/g = 2\pi\bar{a}/\lambda$. Then the boundary-value problem described by (2.3), (2.7), (2.9) and (2.10) becomes

$$\left.\begin{array}{ll}\text{(A)} & \nabla^2 u(x,y,z) = 0 \quad \text{in} \quad y < 0,\\ \text{(B)} & u_y - au = 0 \quad \text{on} \quad y = 0, \text{ outside the disk},\\ \text{(C)} & u_y = x \quad \text{on the disk},\\ \text{(D)} & u(r,\theta,y) - Ar^{-\frac{1}{2}}\exp[ay+iar] = O(1/r) \quad \text{as} \quad r\to\infty,\end{array}\right\} \tag{2.12}$$

where the pressure function u is related to the potential function V by

$$g\bar{a}\Theta_0 au(x,y,z) = i\sigma V(\bar{a}x,\bar{a}y,\bar{a}z). \tag{2.13}$$

3. Integral representation

The solution of the boundary-value problem for a pitching circular disk will be considered. The problem is to find a function $u(r,\theta,y)$ continuous in the region $y \leqslant 0$ such that

$$\left.\begin{array}{ll}(\text{A}^0) & u_{rr}+r^{-1}u_r+r^{-2}u_{\theta\theta}+u_{yy} = 0 \quad \text{in} \quad y < 0,\\ (\text{B}^0) & u_y(r,\theta,0) - au(r,\theta,0) = 0 \quad \text{for} \quad r > 1,\\ (\text{C}^0) & u_y(r,\theta,0) = r\cos\theta \quad \text{for} \quad 0 \leqslant r < 1,\\ (\text{D}^0) & u(r,\theta,y) - Ar^{-\frac{1}{2}}\exp[ay+iar] = O(1/r) \quad \text{as} \quad r\to\infty.\end{array}\right\} \tag{3.1}$$

With a regularity condition at the edge of the disk

$$\lim_{\epsilon\to 0}\iint_{C_\epsilon} uu_n\,dS = 0, \tag{3.2}$$

it has been shown by Peters & Stoker (1957) that the solution of the problem is unique. Here C_ϵ represents a half-tube $(r-1)^2+y^2 = \epsilon$, $y < 0$, surrounding the edge of the disk and n is the normal to the half-tube C_ϵ.

The potential of a source of unit strength necessary for an integral representation of the solution is

$$\begin{aligned} G(r,\rho,\theta,\psi,y) &= R^{-1} - a\,e^{ay}\int_y^\infty e^{-a\eta}(\mu^2+\eta^2)^{-\frac{1}{2}}\,d\eta + i\pi a\,e^{ay}H_0^{(1)}(a\mu)\\ &= R^{-1}+H, \end{aligned} \tag{3.3}$$

where $\mu^2 = r^2+\rho^2-2r\rho\cos(\theta-\psi)$, $R^2 = \mu^2+y^2$, and $H_0^{(1)}(a\mu)$ denotes the Hankel function of order zero. Here the potential G satisfies condition (A^0) of (3.1) for $(r,\theta,y) \neq (\rho,\psi,0)$ and condition (D^0) of (3.1) for ρ bounded. Further,

$$H_y - aH = aR^{-1};$$

hence,
$$G_y - aG = \partial R^{-1}/\partial y. \tag{3.4}$$

We express the potential due to surface distributed sources of strength f as

$$U(r,\theta,y\colon f) = W(r,\theta,y\colon f) + L(r,\theta,y\colon f), \tag{3.5}$$

where
$$W(r,\theta,y\colon f) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 f(\rho,\psi)\,R^{-1}\,d\rho/\rho\,d\psi,$$

$$L(r,\theta,y\colon f) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 f(\rho,\psi)\,H\,d\rho/\rho\,d\psi,$$

f being a continuous function over $0 \leqslant \rho < 1$. Then for any such function f, U satisfies conditions (A^0) and (D^0) of (3.1). Moreover, from (3.4) we obtain

$$U_y - aU = W_y. \tag{3.6}$$

By a theorem of potential theory,

$$\lim_{y \to 0-} W_y(r, \theta, y: f) = 0 \quad \text{if} \quad r > 1,$$
$$= f(r, \theta) \quad \text{if} \quad r < 1. \tag{3.7}$$

Hence, we find that U satisfies condition (B^0) of (3.1). In addition, (3.6) and (3.7) now yield

$$U_y(r, \theta, 0: f) - aU(r, \theta, 0: f) = f(r, \theta) \quad \text{for} \quad 0 \leqslant r < 1. \tag{3.8}$$

Thus, if f is determined from the integral equation

$$f(r, \theta) + a[W(r, \theta, 0: f) + L(r, \theta, 0: f)] = r \cos\theta \quad \text{for} \quad 0 \leqslant r < 1, \tag{3.9}$$

U will also satisfy condition (C^0) of (3.1). Accordingly, U is a solution of the given boundary-value problem.

However, the kernel of the integral equation (3.9) is quite complicated. The given problem will therefore be transformed into a simpler problem using the method developed by MacCamy (1961).

Applying the operator $[-(\partial^2/\partial y^2) - a\,\partial/\partial y]$ to both sides of (3.6) we obtain

$$\left(-\frac{\partial^2}{\partial y^2} + a^2\right)\frac{\partial}{\partial y} U = -\frac{\partial^2}{\partial y^2}\left(\frac{\partial}{\partial y} + a\right) W. \tag{3.10}$$

Since U and W are both harmonic functions of r, θ and y in the lower half-space $y < 0$, we have

$$(\Delta + a^2)\,\partial U/\partial y = \Delta(\partial/\partial y + a)\, W, \tag{3.11}$$

where $\Delta = \partial^2/\partial r^2 + r^{-1}\,\partial/\partial r + r^{-2}\,\partial^2/\partial\theta^2$.

Suppose $U(r, \theta, y: f)$ is a solution of the given problem. Since $r\cos\theta$ is an analytic function in the entire space, then by condition (C^0) of (3.1), U may continue to be harmonic across the disk into $y > 0$. It follows then from (3.11) that W can also be harmonic across the disk. Thus, U and W are analytic for $y \leqslant 0$, $r < 1$. In this case, passing to the limit $y = 0$ in (3.11), we obtain

$$(\Delta + a^2)\, r\cos\theta = \Delta(\partial/\partial y + a)\, W(r, \theta, 0: f). \tag{3.12}$$

If the inverse of the operator Δ is applied to both sides of (3.12) we find

$$W_y(r, \theta, 0: f) + aW(r, \theta, 0: f) = (1 + a^2\Delta^{-1})\, r\cos\theta. \tag{3.13}$$

By (3.7),

$$f(r, \theta) + aW(r, \theta, 0: f) = h(r, \theta), \tag{3.14}$$

where $h(r, \theta) = r\cos\theta + a^2 v$, and $\Delta v = r\cos\theta$. Thus, the kernel of the transformed problem has indeed a simple form.

We consider the solution

$$w(r) = r\int_0^r \rho^{-3}\, d\rho \int_0^\rho \tau^3\, d\tau$$

of an equation

$$\frac{d^2 w}{dr^2} + r^{-1}\frac{dw}{dr} - r^{-2} w = r,$$

and introduce the sum $\bar{h}(r,\theta) = r\cos\theta + a^2 w(r)\cos\theta.$ (3.15)

Further, let the functions $f^0(r)\cos\theta$ and $f^1(r)\cos\theta$ be solutions of the integral equations

$$\left.\begin{aligned} &f^0(r)\cos\theta + aW[r,\theta,0\colon f^0(r)\cos\theta] = \bar{h}(r,\theta),\\ \text{and}\quad &f^1(r)\cos\theta + aW[r,\theta,0\colon f^1(r)\cos\theta] = r\cos\theta.\end{aligned}\right\} \tag{3.16}$$

Then, it can be shown (see MacCamy 1961) that the sum of two potentials with an appropriate constant A, such as

$$U[r,\theta,0\colon f^0\cos\theta] + AU[r,\theta,0\colon f^1\cos\theta] = U[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta],$$

is the solution of the given problem. Now, let us consider how the constant A should be determined. From (3.7) and (3.16) we find

$$(\partial/\partial y + a)\, W[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta] = \bar{h}(r,\theta) + Ar\cos\theta. \tag{3.17}$$

Hence, from (3.11) we obtain

$$(\Delta + a^2)\,\partial U[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta]/\partial y = \Delta[\bar{h}(r,\theta) + Ar\cos\theta]. \tag{3.18}$$

Recalling that $\Delta(r\cos\theta) = 0$, and using (3.15), (3.18) can be expressed as

$$(\Delta + a^2)\,\partial U[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta]/\partial y = (\Delta + a^2)\, r\cos\theta. \tag{3.19}$$

Thus, it follows that $$(\Delta + a^2)\, T(r,\theta) = 0, \tag{3.20}$$

where $$T(r,\theta) = \partial U[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta]/\partial y - r\cos\theta. \tag{3.21}$$

Moreover, the solution of (3.20) is given by $T(r,\theta) = \alpha J_1(ar)\cos\theta$, J_1 being the Bessel function of order one. In order that U satisfy boundary condition (C^0) of (3.1), the constant A should be chosen to make the coefficient α zero. Since $J_1'(0) = \frac{1}{2}$ and $J_1(0) = 0$, the condition

$$\partial T(r,\theta)/\partial r = 0 \quad \text{at} \quad r = 0 \tag{3.22}$$

insures that $\partial U[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta]/\partial y = r\cos\theta.$

Now, from (3.6) and (3.16) we find

$$\begin{aligned}&\partial U[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta]/\partial y\\ &\quad = (\partial/\partial y + a)\, W[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta] + aL[r,\theta,0\colon f^0\cos\theta + Af^1\cos\theta]\\ &\quad = \bar{h}(r,\theta) + Ar\cos\theta + aL[r,\theta,0\colon f^0\cos\theta] + AaL[r,\theta,0\colon f^1\cos\theta].\end{aligned} \tag{3.23}$$

Here, from the previous definitions, (3.3) and (3.5), we have

$$\begin{aligned}L[r,\theta,0\colon f\cos\theta] = -\frac{a}{2\pi}\int_0^{2\pi}\int_0^1 f(\rho)\cos\psi\\ \times\left[\int_0^\infty e^{-a\eta}(\mu^2+\eta^2)^{-\frac{1}{2}}\,d\eta - i\pi H_0^{(1)}(a\mu)\right] d\rho\,\rho\, d\psi,\end{aligned} \tag{3.24}$$

but, as shown in appendix 1, (3.24) can be transformed to

$$\begin{aligned}L[r,\theta,0\colon f\cos\theta] &= -a\int_0^1 \rho f(\rho)\int_0^\infty \frac{J_1(rt)\,J_1(\rho t)}{a+t}\,dt\,d\rho\cos\theta\\ &\quad + i\pi a\left[H_1^{(1)}(ar)\int_0^r \rho f(\rho)\,J_1(a\rho)\,d\rho + J_1(ar)\int_r^1 \rho f(\rho)\,H_1^{(1)}(a\rho)\,d\rho\right]\cos\theta\\ &= l[r,0\colon f]\cos\theta.\end{aligned} \tag{3.25}$$

Substituting (3.23) and (3.25) into (3.21) we write

$$T(r,\theta) = \{a^2w(r)+Ar+al[r,0:f^0]+Aal[r,0:f^1]\}\cos\theta;$$

then by (3.22) the constant A can be determined from the condition

$$\frac{d}{dr}\{a^2w(r)+Ar+al[r,0:f^0]+Aal[r,0:f^1]\}\cos\theta|_{r=0}$$
$$= \{al'[r,0:f^0]|_{r=0}+A+Aal'[r,0:f^1]|_{r=0}\}\cos\theta = 0.$$

Hence
$$A = -\frac{al'[r,0:f^0]|_{r=0}}{1+al'[r,0:f^1]|_{r=0}}. \tag{3.26}$$

Next, we proceed to calculate the derivative of the function l at $r=0$. Since

$$dJ_1(rt)/dr|_{r=0} = \tfrac{1}{2}t,$$

$$\frac{d}{dr}\int_0^1 \rho f(\rho)\int_0^\infty \frac{J_1(rt)\,J_1(\rho t)}{a+t}\,dt\,d\rho|_{r=0} = \frac{1}{2}\int_0^1 \rho f(\rho)\int_0^\infty \frac{tJ_1(\rho t)}{a+t}\,dt\,d\rho.$$

Further, we find
$$\frac{d}{dr}\left[H_1^{(1)}(ar)\int_0^r \rho f(\sigma)\,J_1(a\rho)\,d\rho\right]\Bigg|_{r=0} = 0,$$

and
$$\frac{d}{dr}\left[J_1(ar)\int_r^1 \rho f(\rho)\,H_1^{(1)}(a\rho)\,d\rho\right]\Bigg|_{r=0} = \tfrac{1}{2}a\int_0^1 \rho f(\rho)H_1^{(1)}(a\rho)\,d\rho.$$

Hence, we obtain the derivative of (3.25)

$$l'[r,0:f]|_{r=0} = -\tfrac{1}{2}a\int_0^1 \rho f(\rho)\int_0^\infty \frac{t}{a+t}J_1(\rho t)\,dt\,d\rho+\tfrac{1}{2}i\pi a^2\int_0^1 \rho f(\rho)\,H_1^{(1)}(a\rho)\,d\rho. \tag{3.27}$$

Now, by the identity (Gröbner & Hofreiter 1961),

$$\int_0^\infty \frac{x}{x+k}J_1(ax)\,dx = -\tfrac{1}{2}\pi k[S_{-1}(ak)-Y_{-1}(ak)],$$

(3.27) can be written in the simple form

$$l'[r,0:f]|_{r=0} = \tfrac{1}{4}\pi a^2\int_0^1 \rho f(\rho)[S_{-1}(a\rho)-Y_{-1}(a\rho)]\,d\rho+\tfrac{1}{2}i\pi a^2\int_0^1 \rho f(\rho)H_1^{(1)}(a\rho)\,d\rho, \tag{3.28}$$

where S_{-1} and Y_{-1} denote the Struve function and Neumann function of order -1. Lastly, the relations among various functions,

$$S_{-1}(z) = W_1(z), \quad Y_{-1}(z) = -Y_1(z),$$

and
$$H_1^{(1)}(z) = J_1(z)+iY_1(z),$$

where W_1 is the Weber function of order 1, permit us to write

$$l'[r,0:f]|_{r=0} = \tfrac{1}{4}\pi a^2\int_0^1 \rho f(\rho)\,[W_1(a\rho)-Y_1(a\rho)+i2J_1(a\rho)]\,d\rho. \tag{3.29}$$

The substitution of (3.29) into (3.26) yields

$$A = -\frac{\tfrac{1}{4}\pi a^3\int_0^1 \rho f^0(\rho)[W_1(a\rho)-Y_1(a\rho)+i2J_1(a\rho)]\,d\rho}{1+\tfrac{1}{4}\pi a^3\int_0^1 \rho f^1(\rho)\,[W_1(a\rho)-Y_1(a\rho)+i2J_1(a\rho)]\,d\rho}. \tag{3.30}$$

Thus, we have reduced the solution of the given boundary-value problem to the solution of (3.16). With known values of f^0 and f^1, the constant A can be evaluated by (3.30). Finally, we obtain the solution in the form

$$u(r,\theta,y) = U[r,\theta,y: f(r,\theta)] = U(r,\theta,y: f^0\cos\theta + Af^1\cos\theta). \tag{3.31}$$

4. Added moment of inertia and damping coefficient

We relate the moment acting on the pitching disk to the quantities called the added moment of inertia and damping coefficient. Then we shall consider the limiting values of these quantities when the frequency parameter a tends to zero.

The moment acting on the disk due to the dynamic fluid pressure is given by

$$G(t) = \iint_S (-\rho^*\Phi_t)\,\bar{x}\cos(n,\bar{y})\,dS. \tag{4.1}$$

For small oscillatory motions, the linearization (see John 1950) yields the first-order moment as

$$G^1(t) = \iint_{S^0(\bar{x},\bar{z})} (-\rho^*\Phi_t)\,\bar{x}\,dS,$$

where S represents the immersed surface of the disk in motion, while S^0 denotes that in the undisturbed position. From (2.2) and (2.13) we then have

$$G^1(t) = \rho^* g a^{-4}\Theta_0 \,\mathrm{Re}\iint_{S^0(x,z)} x\,au\,e^{-i\sigma t}dS. \tag{4.2}$$

Since the angular velocity and the angular acceleration of the pitching motion are given by

$$\dot{\Theta} = \mathrm{Re}[-i\sigma\Theta_0 e^{-i\sigma t}], \quad \ddot{\Theta} = \mathrm{Re}[-\sigma^2\Theta_0 e^{-i\sigma t}],$$

if we express the first-order moment as

$$G^1(t) = -\bar{I}\bar{a}\ddot{\Theta} - \bar{H}\bar{a}\dot{\Theta}, \tag{4.3}$$

we obtain

$$\frac{\bar{I}}{\rho^*\bar{a}^4} = \frac{1}{a}\mathrm{Re}\left\{\iint_{S^0} x[au(x,0,z)]\,dS\right\}, \tag{4.4}$$

and

$$\frac{\bar{H}}{\rho^*\bar{a}^4\sigma} = \frac{1}{a}\mathrm{Im}\left\{\iint_{S^0} x[au(x,0,z)]\,dS\right\}. \tag{4.5}$$

Here $\bar{I}$ and $\bar{H}$ are called the added moment of inertia and the damping coefficient, respectively. Hence, in terms of polar co-ordinates we have

$$I = \frac{\bar{I}}{\rho^*\bar{a}^4} = \frac{1}{a}\mathrm{Re}\left\{\int_0^{2\pi}\int_0^1 r\cos\theta[au(r,\theta,0)]\,r\,dr\,d\theta\right\}, \tag{4.6}$$

$$H = \frac{\bar{H}}{\rho^*\bar{a}^4\sigma} = \frac{1}{a}\mathrm{Im}\left\{\int_0^{2\pi}\int_0^1 r\cos\theta[au(r,\theta,0)]\,r\,dr\,d\theta\right\}, \tag{4.7}$$

where we find from (3.8) that the pressure function au is related to the density $f(r,\theta)$ of the distributed sources by

$$\begin{aligned} au(r,\theta,0) &= r\cos\theta - f(r,\theta) \\ &= r\cos\theta - \mathrm{Re}[f(r,\theta)] - i\,\mathrm{Im}[f(r,\theta)]. \end{aligned} \tag{4.8}$$

When $y = 0$, by the identity (Gröbner & Hofreiter 1961)

$$\int_0^\infty e^{-a\eta}(\mu^2+\eta^2)^{-\frac{1}{2}}\,d\eta = \tfrac{1}{2}\pi[S_0(a\mu)-Y_0(a\mu)],$$

the potential of a source of unit strength (3.3) can be evaluated explicitly as

$$G(r,\rho,\theta,\psi\colon a) = \mu^{-1}-\tfrac{1}{2}\pi a[Y_0(a\mu)+S_0(a\mu)-i2J_0(a\mu)]. \tag{4.9}$$

The functions appearing in (4.9) have expansions of the following form

$$J_0(a\mu) = \sum_{m=0}^{\infty} A_m(\mu)\,a^{2m}, \quad S_0(a\mu) = \frac{2}{\pi}\sum_{m=0}^{\infty} B_m(\mu)\,a^{2m+1},$$

and $$Y_0(a\mu) = \frac{2}{\pi}\left[\log a\sum_{m=0}^{\infty} A_m(\mu)\,a^{2m}+\sum_{m=0}^{\infty} C_m(\mu)\,a^{2m}\right],$$

where $$A_0 = 1, A_1 = -\mu^2/2^2,\ldots, B_0 = \mu, \quad B_1 = -\mu^3/1\cdot 3,\ldots,$$

$$C_0 = \gamma+\log\tfrac{1}{2}\mu, \quad C_1 = -(\gamma+\log\tfrac{1}{2}\mu-1)\,\mu^2/2^2,\ldots.$$

Rearranging power series of parameter a we obtain

$$aG(r,\rho,\theta,\psi\colon a) = \sum_{n=0}^{\infty} P_n a^{n+1}+\sum_{n=0}^{\infty} Q_n a^{n+2}\log a, \tag{4.10}$$

where $$P_0 = 1/\mu, \quad P_1 = i\pi-\gamma-\log\tfrac{1}{2}\mu, \quad P_2 = -\mu,\ldots,$$

$$Q_0 = -1, \quad Q_1 = 0, \quad Q_2 = \mu^2/2^2,\ldots.$$

We shall, therefore, develop the asymptotic solution of the integral equation (3.9) which depends upon parameter a in the form

$$f(r,\theta) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij}\,a^i(a\log a)^j. \tag{4.11}$$

As shown in appendix 2, it turns out that

$$f(r,\theta\colon a) \sim f_{00}+af_{10}+a(a\log a)f_{11}+a^2f_{20}+a^2(a\log a)f_{21}+a^3f_{30}, \tag{4.12}$$

where

$$f_{00} = r\cos\theta, \quad f_{10} = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{\rho\cos\psi}{\mu}\,d\rho\,\rho\,d\psi,$$

$$f_{11} = 0,$$

$$f_{20} = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\left[(i\pi-\gamma-\log\tfrac{1}{2}\mu)\,\rho\cos\psi-\frac{1}{2\pi\mu}\int_0^{2\pi}\int_0^1\frac{\rho\cos\psi}{\mu}\,d\rho\,\rho\,d\psi\right]d\rho\,\rho\,d\psi,$$

$$f_{21} = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\left(\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\frac{\rho\cos\psi}{\mu}\,d\rho\,\rho\,d\psi\right)d\rho\,\rho\,d\psi,$$

and

$$f_{30} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1\left\{\mu\rho\cos\psi+(i\pi-\gamma-\log\tfrac{1}{2}\mu)\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\frac{\rho\cos\psi}{\mu}\,d\rho\,\rho\,d\psi\right.$$
$$\left.+\frac{1}{\mu}\left[\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\left(-\log\tfrac{1}{2}\mu\,\rho\cos\psi+\frac{1}{2\pi\mu}\int_0^{2\pi}\int_0^1\frac{\rho\cos\psi}{\mu}\,d\rho\,\rho\,d\psi\right)d\rho\,\rho\,d\psi\right]\right\}d\rho\,\rho\,d\psi.$$

Substituting the solution (4.12) in (4.6) and (4.7) we obtain the following first non-vanishing terms,

$$I(0) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \left[\int_0^{2\pi}\int_0^1 \frac{1}{\mu}\rho\cos\psi\, d\rho\, \rho\, d\psi\right] r\cos\theta\, dr\, r\, d\theta, \qquad (4.13)$$

and $$H(0) = -\frac{a^2}{4\pi}\int_0^{2\pi}\int_0^1 \left[\int_0^{2\pi}\int_0^1 \frac{1}{\mu}\rho\cos\psi\, d\rho\, \rho\, d\psi\right] r\cos\theta\, dr\, r\, d\theta = O(a^2), \quad (4.14)$$

where $\mu^2 = r^2+\rho^2-2r\rho\cos(\theta-\psi)$. Note that as the parameter a tends to zero, the normalized damping coefficients H vanishes while the normalized added moment of inertia I approaches a non-zero constant.

5. Numerical procedure

We present here a numerical method for finding the approximate value of the unknown function $f^j(r,\theta) = f^j(r)\cos\theta$, $j = 0, 1$, in the integral equation (3.16). Presently, a scheme of evaluating the limiting value of the normalized added moment of inertia will also be shown.

Let us begin by writing (3.16) as

$$f^j(r)\cos\theta + aW[r,\theta,0; f^j(r)\cos\theta] = g^j(r)\cos\theta \quad (0 \leqslant r < 1), \qquad (5.1)$$

where the integral W is given by

$$W(r,\theta,0: f) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 f(\rho)\cos\psi\,\mu^{-1}\, d\rho\, \rho\, d\psi, \qquad (5.2)$$

$$\mu^2 = r^2+\rho^2-2r\rho\cos(\theta-\psi).$$

First, setting $\alpha = r^2+\rho^2$, $\beta = 2r\rho$, and $\psi-\theta = \gamma$, we evaluate the ψ-integration in (5.2):

$$\int_0^{2\pi}\cos\psi\,\mu^{-1}d\psi = 2\cos\theta\int_0^{\pi}\cos\gamma\,[\alpha-\beta\cos\gamma]^{-\frac{1}{2}}\, d\gamma.$$

Then by (A 1.3) in appendix 1, we find

$$\int_0^{2\pi}\cos\psi\,\mu^{-1}d\psi = 4\cos\theta(\alpha+\beta)^{-\frac{1}{2}}\left\{\frac{\alpha}{\beta}\left[K\left(\frac{2\beta}{\alpha+\beta}\right)^{\frac{1}{2}} - E\left(\frac{2\beta}{\alpha+\beta}\right)^{\frac{1}{2}}\right] - E\left(\frac{2\beta}{\alpha+\beta}\right)^{\frac{1}{2}}\right\}$$

$$= \frac{2\cos\theta}{r\rho}\left[\frac{\rho^2+r^2}{\rho+r}K(k) - (\rho+r)E(k)\right], \qquad (5.3)$$

where k denotes the modulus of Jacobian elliptic functions given by

$$k^2 = 4r\rho/(\rho+r)^2.$$

Substitution of (5.3) into (5.1) yields the integral equation relating boundary values only on the radius of the disk,

$$f^j(r) + \frac{a}{\pi}\int_0^1 \frac{f^j(\rho)}{r}\left[\frac{\rho^2+r^2}{\rho+r}K(k) - (\rho+r)E(k)\right]d\rho = g^j(r) \quad (0 \leqslant r < 1). \quad (5.4)$$

We observe that the integral term will vanish for $r = 0$, hence $f^j(0) = g^j(0)$, and for $\rho = 0$ the integrand becomes zero. Further, at $\rho = r$, the kernel of this equation is logarithmically singular since

$$\lim_{k\to 1}\left[K(k) - \ln\frac{4}{(1-k^2)^{\frac{1}{2}}}\right] = 0. \qquad (5.5)$$

Let us introduce a regular function R by

$$R(k) = K(k) + \ln(1-k^2)^{\frac{1}{2}}, \tag{5.6}$$

then (5.4) may be written as

$$f^j(r) - \frac{a}{\pi}\int_0^1 \frac{f^j(\rho)}{r}\frac{\rho^2+r^2}{\rho+r}\ln|\rho-r|\,d\rho + \frac{a}{\pi}\int_0^1 \frac{f^j(\rho)}{r} \times \left\{\frac{\rho^2+r^2}{\rho+r}[\ln(\rho+r)+R(k)] - (\rho+r)E(k)\right\}d\rho = g^j(r). \tag{5.7}$$

Here our procedure is to divide the interval (0, 1) of the first integral of (5.7) into the subregion about the point of discontinuity $(r-h, r+h)$, and for the remaining region we write

$$J(r) = \int_0^1 \frac{f^j(\rho)}{r}\frac{\rho^2+r^2}{\rho+r}\ln|\rho-r|\,d\rho = \int_0^{r-h} + \int_{r-h}^{r+h} + \int_{r+h}^1 \frac{f^j(\rho)}{r}\frac{\rho^2+r^2}{\rho+r}\ln|\rho-r|\,d\rho. \tag{5.8}$$

Now we make use of the following approximate quadrature formula for $r \neq 1$ which would give exact results if $F(\rho)$ were polynomials of degree less than or equal to three, i.e.

$$\int_{r-h}^{r+h} F(\rho)\ln|\rho-r|\,d\rho \approx \tfrac{1}{3}h(\ln h - \tfrac{1}{3})[F(r-h)+F(r+h)] + \tfrac{4}{3}h(\ln h - \tfrac{4}{3})F(r),$$

and for $r = 1$ a similar quadrature formula

$$\int_{1-2h}^{1} F(\rho)\ln(1-\rho)\,d\rho \approx \tfrac{1}{3}h(\ln 2h + \tfrac{1}{6})F(1-2h) + \tfrac{4}{3}h(\ln 2h - \tfrac{5}{6})F(1-h) + \tfrac{1}{3}h(\ln 2h - \tfrac{17}{6})F(1).$$

For example, in the case of $r \neq 1$, we find the approximation of the integral over the subregion as

$$S(r) = \int_{r-h}^{r+h} \frac{f^j(\rho)}{r}\frac{\rho^2+r^2}{\rho+r}\ln|\rho-r|\,d\rho \approx \tfrac{4}{3}h(\ln h - \tfrac{4}{3})f^j(r) + \tfrac{1}{3}h(\ln h - \tfrac{1}{3})\left[\frac{2r^2-h(2r-h)}{r(2r-h)}f^j(r-h) + \frac{2r^2+h(2r+h)}{r(2r+h)}f^j(r+h)\right]. \tag{5.9}$$

Then the substitution of (5.8) and (5.9) in (5.7) yields

$$\left[1 - \frac{4ah}{3\pi}(\ln - \tfrac{4}{3})\right]f^j(r) - \frac{ah}{3\pi}(\ln h - \tfrac{1}{3})\left[\frac{2r^2-h(2r-h)}{r(2r-h)}f^j(r-h) + \frac{2r^2+h(2r+h)}{r(2r+h)}f^j(r+h)\right] - \frac{a}{\pi}\int_0^{r-h}\int_{r+h}^1 \frac{f^j(\rho)}{r}\frac{\rho^2+r^2}{\rho+r}\ln|\rho-r|\,d\rho + \frac{a}{\pi}\int_0^1 \frac{f^j(\rho)}{r}\left\{\frac{\rho^2+r^2}{\rho+r}[\ln(\rho+r)+R(k)] - (\rho+r)E(k)\right\}d\rho = g^j(r) \quad (0 \leqslant r < 1). \tag{5.10}$$

We remark here that within the subregion, the coefficient of the logarithmic term is assumed to be a polynomial of degree equal to or less than three. This permits the use of a larger interval size than would be possible if the coefficient were assumed to be constant within the subregion (as is often done). In (5.10), the remaining integrals are readily approximated by application of Simpson's and the trapezoidal rules. Further, we observe that the unknown function $f^j(r)$ is a complex function. However, the kernel of the integral equation is real, and the imaginary part of the given function $\operatorname{Im}[g^j(r)]$ is zero, hence we have $\operatorname{Im}[f^j(r)] = 0$. Here, the real parts of $g^j(r)$ are given by $\operatorname{Re}[g^0(r)] = r + \frac{1}{8}a^2r^3$ and $\operatorname{Re}[g^1(r)] = r$, respectively.

From (4.13) and (5.3), as the parameter a tends to zero, we have

$$I(0) = \frac{1}{\pi}\int_0^{2\pi}\int_0^1\left\{\int_0^1 \frac{\rho}{r}\left[\frac{\rho^2+r^2}{\rho+r}K(k)-(\rho+r)E(k)\right]d\rho\right\} r^2\cos^2\theta\,dr\,d\theta$$

$$= \int_0^1 r^2\left\{\int_0^1 \frac{\rho}{r}\left[\frac{\rho^2+r^2}{\rho+r}K(k)-(\rho+r)E(k)\right]d\rho\right\}dr. \tag{5.11}$$

Note that for $r = 0$ the integral becomes zero, and for $\rho = 0$ the integrand of (5.11) vanishes. Now, by use of the regular function R, (5.11) may be expressed in the form

$$I(0) = \int_0^1 r^2\left\{-\int_0^1 \frac{\rho(\rho^2+r^2)}{r(\rho+r)}\ln|\rho-r|\,d\rho + \int_0^1 \frac{\rho(\rho^2+r^2)}{r(\rho+r)}[\ln(\rho+r)+R(k)]\,d\rho - \int_0^1 \frac{\rho(\rho+r)}{r}E(k)\,d\rho\right\}dr. \tag{5.12}$$

The limiting value of I can therefore be evaluated in a manner quite similar to that used for evaluating the integral term in (5.4).

Next, let us examine the behaviour of the integrand of (3.30) as ρ approaches zero. It can be shown that

$$\lim_{\rho\to 0}[\rho W_1(a\rho)] = 0, \quad \lim_{\rho\to 0}[\rho Y_1(a\rho)] = \frac{2}{\pi a}, \quad \text{and} \quad \lim_{\rho\to 0}[\rho J_1(a\rho)] = 0.$$

Since $f^j(0) = g^j(0) = 0$, the entire integrand vanishes when $\rho = 0$, i.e.

$$\lim_{\rho\to 0}\{\rho f^j(\rho)\,[W_1(a\rho)-Y_1(a\rho)+i2J_1(a\rho)]\} = 0.$$

Furthermore, we observe that as the argument $a\rho$ becomes large,

$$W_1(a\rho) \approx -Y_1(a\rho),$$

hence

$$\lim_{a\rho\to\infty}[W_1(a\rho)-Y_1(a\rho)+i2J_1(a\rho)] = -2\left[\frac{\sin(a\rho-\frac{3}{4}\pi)}{(\pi a\rho/2)^{\frac{1}{2}}} - i\frac{\cos(a\rho-\frac{3}{4}\pi)}{(\pi a\rho/2)^{\frac{1}{2}}}\right]$$

$$= i2\,(2/\pi a\rho)^{\frac{1}{2}}\,e^{i(a\rho-\frac{3}{4}\pi)}. \tag{5.13}$$

Therefore, this term in the integrand of (3.30) fluctuates as the parameter a increases. This implies that it is necessary to take a finer interval for large value of a in order to obtain the same accuracy in the numerical results.

Let us designate the real and imaginary parts of the denominator of (3.30) by Re D and Im D, and those of the numerator by Re N and Im N, then we have

$$\operatorname{Re} A = \frac{(\operatorname{Re} D)(\operatorname{Re} N) + (\operatorname{Im} D)(\operatorname{Im} N)}{(\operatorname{Re} N)^2 + (\operatorname{Im} N)^2} \quad \text{and} \quad \operatorname{Im} A = \frac{(\operatorname{Re} D)(\operatorname{Im} N) - (\operatorname{Im} D)(\operatorname{Re} N)}{(\operatorname{Re} N)^2 + (\operatorname{Im} N)^2}. \tag{5.14}$$

The computational procedure is to divide the interval of integration (0, 1) into as many equal parts as practical. Then, at each grid point the integral terms in (5.10) will be evaluated by Simpson's rule to form an approximate system of linear equations relating the values of $f^j(r)$ at selected grid points. Solving the linear equations, we can determine the unknown $f^j(r)$. Then we proceed to evaluate the real and imaginary parts of the coefficient A by (5.14) using Simpson's rule. Thus, the solution of the integral equation

$$\begin{aligned} f(r,\theta) &= [f^0(r) + Af^1(r)]\cos\theta \\ &= [f^0(r) + (\operatorname{Re} A)f^1(r)]\cos\theta + i(\operatorname{Im} A)f^1(r)\cos\theta \\ &= \operatorname{Re}[f(r,\theta)] + i\operatorname{Im}[f(r,\theta)] \end{aligned} \tag{5.15}$$

can be determined approximately. Now, with the values of the real and imaginary parts of $f(r,\theta)$ at hand, we can evaluate the dynamic pressure at the grid points by use of (4.8),

$$\begin{aligned} p^0 = au(r,\theta,0) &= [r - f^0(r) - (\operatorname{Re} A)f^1(r)]\cos\theta - i(\operatorname{Im} A)f^1(r)\cos\theta \\ &= \operatorname{Re} p^0 + i\operatorname{Im} p^0 \end{aligned} \tag{5.16}$$

and its intensity by
$$|p^0| = [(\operatorname{Re} p^0)^2 + (\operatorname{Im} p^0)^2]^{\frac{1}{2}}. \tag{5.17}$$

In addition, the phase lag of the pressure is

$$\epsilon = \tan^{-1}[\operatorname{Im} p^0/\operatorname{Re} p^0]. \tag{5.18}$$

It follows, then, from (4.6) and (4.7) that the normalized added moment of inertia and damping coefficient can be evaluated from

$$I = \frac{\pi}{a}\int_0^1 r^2[r - f^0(r) - (\operatorname{Re} A)f^1(r)]\,dr, \tag{5.19}$$

$$H = -\frac{\pi}{a}\int_0^1 r^2(\operatorname{Im} A)f^1(r)\,dr. \tag{5.20}$$

6. Results

A numerical method of solving the integral equation (3.9) of two variables was considered in an earlier paper (Kim 1962). Using the numerical solution of (3.9), the normalized added moment of inertia I, and the normalized damping coefficient H of a pitching circular disk were evaluated as a function of the frequency parameter $a = \sigma^2\bar{a}/g$. The numerical scheme was based on establishing a suitable lattice on the circular disk (and the elliptic disk) lying on the (x, z)-plane for the finite difference representation of the integral equation. The x-axis of the disk was divided into sixteen equal intervals h, while the ordinates parallel to the z-axis were divided into eight equal intervals $k(x)$; thus each point on the lattice has the co-ordinates

$$\left.\begin{aligned} x_i &= (i-8)\,h \qquad (i = 0, 1, \ldots, 16, \text{ where } h = \tfrac{1}{8}), \\ z_j(x_i) &= (j-4)\,k(x_i) \quad (j = 0, 1, \ldots, 8, \text{ where } k(x_i) = \tfrac{1}{4}(1 - x_i^2)^{\frac{1}{2}}). \end{aligned}\right\} \tag{6.1}$$

Eighty-two linear equations were required to relate the values of Re $[f(x,z)]$ and Im $[f(x,z)]$ in (3.9) at forty-one pivotal points (located in one quadrant of the lattice). For reasonably large values of a, however, the function G given by (3.3) is approximated by

$$G(x,0,z,\xi,0,\zeta) \sim r^{-1} + i(2\pi a/r)^{\frac{1}{2}}\, e^{i(ar-\frac{1}{4}\pi)}, \tag{6.2}$$

where $r^2 = (x-\xi)^2+(z-\zeta)^2$, and the quadrature of such an oscillatory function requires a quite small finite-difference interval. If the original intervals (6.1) be bisected, there results 290 equations for 145 pivotal points.

The present study was undertaken chiefly to avoid the need for handling such a large number of linear equations in order to obtain accurate numerical results up to $a \approx 4{\cdot}0$, which covers the physically significant range of the frequency parameter. Furthermore, the new results may be used to ascertain the validity of the numerical scheme adopted previously for obtaining the solution of (3.9).

Here we begin by dividing the radius of the disk into thirty-two equal parts so that four times as many intervals as in the case of two variables are obtained on the radius. The integral equations (3.16) are then replaced by two finite sets of linear equations relating the unknowns f^0 and f^1 at each pivotal point. After the numerical solutions of these linear equations have been found by the elimination process based on the algorithm of Gauss, the real and imaginary parts of the constant A can be evaluated using (3.30) and (5.14). The term (5.13) in the integrand of (3.30), however, possesses an oscillatory tendency as the parameter a increases. Nevertheless, in the present method a finite-difference interval can readily be made smaller without excessively increasing the number of linear equations because the quadrature depends upon only one variable. The alternative method of finding the solution of the integral equations (3.16) would be to use the iteration process suitable for the equation with logarithmic kernel described by Wagner (1951).

The computation is performed for ten values of a, i.e. $\frac{1}{12}\pi$, $\frac{1}{6}\pi$, $\frac{1}{5}\pi$, $\frac{1}{4}\pi$, $\frac{1}{3}\pi$, $\frac{2}{5}\pi$, $\frac{1}{2}\pi$, $\frac{2}{3}\pi$, π, and $\frac{5}{4}\pi$. By the definition of the frequency parameter, the minimum and maximum values of a correspond to the cases in which the length of the wave generated by the forced pitching of the disk is equal to 12 to 0·8 times the diameter, respectively. Using computed values of the source density along the radius of the disk we can evaluate I and H by (5.19) and (5.20)

We remark here that the accuracy of the computation can be checked by evaluating H by a formula (A 3.13) in appendix 3. For the evaluation of H by (5.20) only the imaginary part of the pressure acting on the surface of the disk is necessary. However, in the formula (A 3.13), in order to consider the energy of radiating waves at large distances, we require both real and imaginary parts of the pressure. In table 1, the values of H evaluated by these two methods are presented. The comparison of results indicates the good accuracy of the present computation.

In figures 1 and 2, the dependence of the normalized added moment of inertia and normalized damping coefficient on the frequency parameter are presented. The solid lines represent the results computed by the present method, and the circles indicate the results obtained by the previous method (the method using (3.9)). The values of I obtained by the previous method are somewhat lower

a	H evaluated by (5·20)	H evaluated by (A 3.13)
$\frac{1}{12}\pi$	0·0023	0·0023
$\frac{1}{6}\pi$	0·0138	0·0138
$\frac{1}{5}\pi$	0·0210	0·0210
$\frac{1}{4}\pi$	0·033	0·033
$\frac{1}{3}\pi$	0·054	0·054
$\frac{2}{5}\pi$	0·068	0·068
$\frac{1}{2}\pi$	0·083	0·083
$\frac{2}{3}\pi$	0·096	0·096
π	0·097	0·098
$\frac{5}{4}\pi$	0·093	0·095

TABLE 1

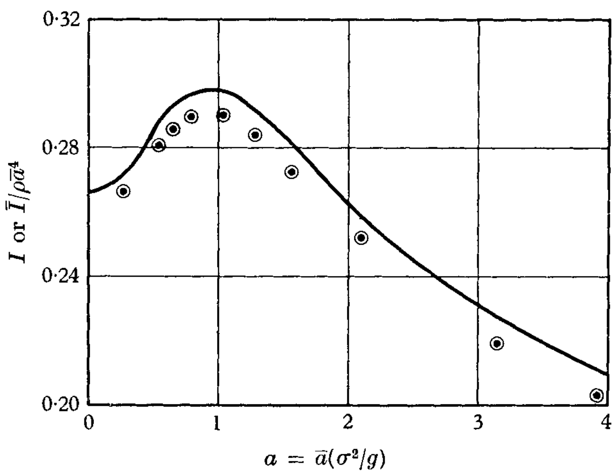

FIGURE 1. The dependence of the normalized moment of inertia on the frequency parameter.

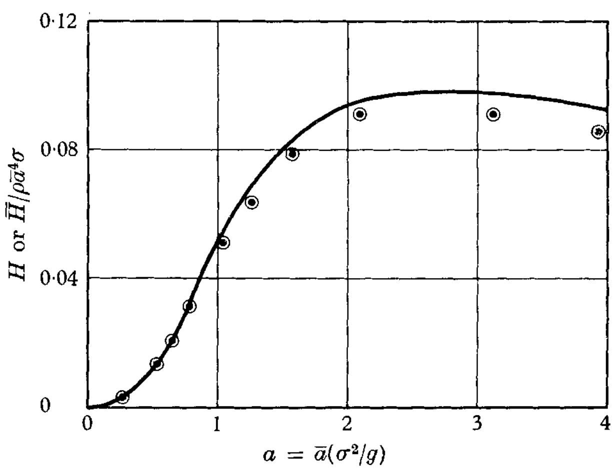

FIGURE 2. The dependence of the normalized damping coefficient on the frequency parameter.

throughout the range of the computation, while the values of H obtained by the previous method are in good agreement with the present results up to $a \approx 1{\cdot}0$, and then tend to lower values. These facts indicate that to improve the accuracy of the results in the method using (3.9) finer finite-difference intervals must be taken.

The limiting value of the normalized added moment of inertia, as a tends to zero (computed by (5.12)), is $I(0) = 0{\cdot}266$.

Next, we present the strip-method evaluation for the circular disk problem by making use of the data from MacCamy's unpublished note, 'The rolling motion of strip'. Let us denote the dynamic pressures of the two-dimensional raft problem and the three-dimensional circular disk problem for a given value of the parameter a by

$$\pi_2(a) = -\rho^* \partial\Phi_2(\bar{x}, 0, t;\, a)/\partial t \quad \text{and} \quad \pi_3(a) = -\rho^* \partial\Phi_3(\bar{x}, 0, \bar{z}, t;\, a)/\partial t.$$

The half-width of a circular disk of radius $\bar{a}$ at distance $\bar{z}$ from the centre is $w(\bar{z}) = (\bar{a}^2 - \bar{z}^2)^{\frac{1}{2}}$, hence we have

$$kw(\bar{z}) = (a/\bar{a})\,(\bar{a}^2 - \bar{z}^2)^{\frac{1}{2}} = a(1 - \bar{z}^2/\bar{a}^2)^{\frac{1}{2}}. \tag{6.3}$$

Over the strip $w(\bar{z})\,d\bar{z}$, if the pressure $\pi_3(a)$ is approximated by $\pi_2\{a(1-z^2/a^2)^{\frac{1}{2}}\}$, the approximate moment acting on the pitching disk of small draft is given by

$$\begin{aligned} G^* &= \int_{-\bar{a}}^{\bar{a}} \int_{-w(\bar{z})}^{w(\bar{z})} \pi_2\{a(1 - \bar{z}^2/\bar{a}^2)^{\frac{1}{2}}\}\, \bar{x}\, d\bar{x}\, d\bar{z} \\ &= 4\int_0^{\bar{a}} \int_0^{w(\bar{z})} \pi_2\{a(1 - \bar{z}^2/\bar{a}^2)^{\frac{1}{2}}\}\, \bar{x}\, d\bar{x}\, d\bar{z}. \end{aligned} \tag{6.4}$$

For the raft problem, the moment is related to the two-dimensional normalized added moment of inertia and normalized damping coefficient by

$$2\int_0^l \pi_2(a^*) x\, dx = -[\rho^* l^3 I_2(a^*)]\, l\ddot{\theta} - [\rho^* l^3 \sigma H_2(a^*)]\, l\dot{\theta}, \tag{6.5}$$

where $kl = a^*$. Hence, by (6.3), the relation (6.5) can be applied to (6.4) to yield

$$G^* = -2\rho^* \int_0^{\bar{a}} (\bar{a}^2 - \bar{z}^2)^2 I_2\{a(1 - \bar{z}^2/\bar{a}^2)^{\frac{1}{2}}\}\, d\bar{z}\, \ddot{\theta} - 2\rho^* \sigma \int_0^{\bar{a}} (\bar{a}^2 - \bar{z}^2)^2 H_2\{a(1 - \bar{z}^2/\bar{a}^2)^{\frac{1}{2}}\}\, d\bar{z}\, \dot{\theta}. \tag{6.6}$$

Then, by (4.3), (4.6) and (4.7), we may equate the approximate moment to the three-dimensional normalized added moment of inertia and normalized damping coefficient for the circular disk problem as

$$G^* = -[\rho^* \bar{a}^4 I_3(a)]\, \bar{a}\ddot{\theta} - [\rho^* \bar{a}^4 \sigma H_3(a)]\, \bar{a}\dot{\theta}. \tag{6.7}$$

From (6.6) and (6.7) we obtain

$$I_3(a) = 2\int_0^1 (1 - z^2)^2 I_2\{a(1 - z^2)^{\frac{1}{2}}\}\, dz \tag{6.8}$$

and

$$H_3(a) = 2\int_0^1 (1 - z^2)^2 H_2\{a(1 - z^2)^{\frac{1}{2}}\}\, dz, \tag{6.9}$$

where z represents $\bar{z}/\bar{a}$.

The computed values of I_3 and H_3 using MacCamy's data (I_2 and H_2) are compared with the present results in table 2. It can be seen that the strip-method evaluation gives higher values than the more accurate present method throughout the range of the computation.

a	I	I_3	H	H_3
$\frac{1}{6}\pi$	0·288	0·368	0·0138	0·0534
$\frac{1}{3}\pi$	0·298	0·349	0·054	0·107
$\frac{1}{2}\pi$	0·281	0·308	0·083	0·129
$\frac{2}{3}\pi$	0·260	0·281	0·096	0·126
π	0·227	0·245	0·099	0·116

TABLE 2

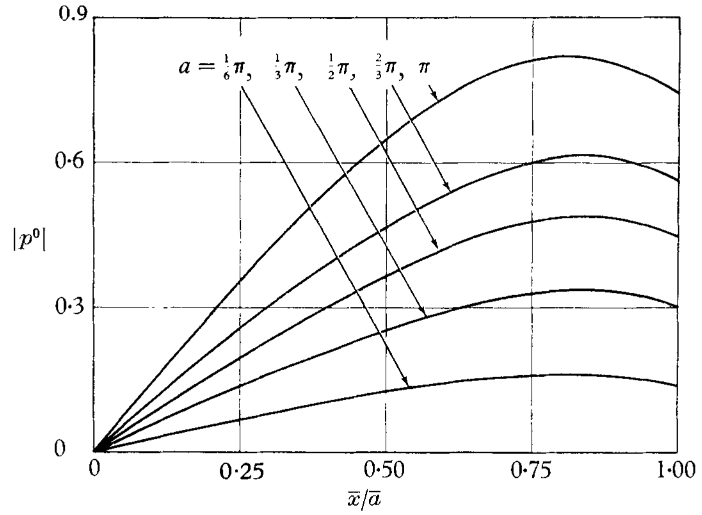

FIGURE 3. The pressure intensity along the axis perpendicular to the axis of rotation.

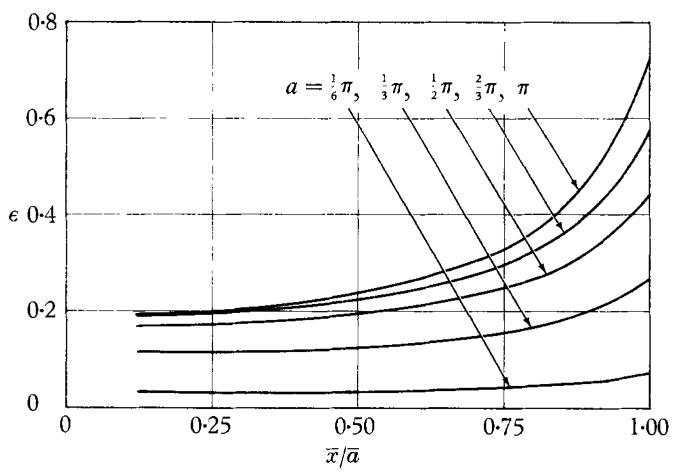

FIGURE 4. The phase lag of pressure along the axis perpendicular to the axis of rotation.

The pressure distribution and the phase lag of the pressure for values of $a = \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi$ and π along the positive x-axis (perpendicular to the axis of rotation of the disk) are shown in figures 3 and 4, respectively. The pressure along the axis of rotation is zero, and it reaches a maximum at about eight-tenths radius. At the edge on the x-axis, the pressure has a finite value so that it will create a wave disturbance there. The pressure intensity increases as the frequency increases. It should be noted that the pressure distribution in the angular direction, as has been assumed in the analysis, depends only on $\cos\theta$.

For a sufficiently large value of a, however, Holford found that the pressure along the x-axis is given by

$$p(r) \approx (4a/3\pi)\,r(1-r^2)^{\frac{1}{2}} \quad (0 \leqslant r < 1). \tag{6.10}$$

It follows then that the maximum pressure will occur at $1/\sqrt{2}$ radius, and at the edge it will vanish.

In figure 4, a sharp rise in the phase lag can be noted between the three-quarter radius and the full radius. The slope of this rise increases with increasing values of the frequency.

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Appendix 1

The integral $L(r,\theta,y\colon f)$ is defined as

$$L(r,\theta,y\colon f) = -\frac{a}{2\pi}\int_0^{2\pi}\int_0^1 f(\rho,\psi)\left[e^{ay}\int_y^\infty e^{-a\eta}(\mu^2+\eta^2)^{-\frac{1}{2}}\,d\eta - i\pi e^{ay}H_0^{(1)}(a\mu)\right]d\rho\,\rho\,d\psi;$$

hence, for $y = 0$, and $f(r,\theta) = f(r)\cos\theta$, we have

$$L[r,\theta,0\colon f(r)\cos\theta]$$
$$= -\frac{a}{2\pi}\int_0^{2\pi}\int_0^1 f(\rho)\cos\psi\left[\int_0^\infty e^{-a\eta}(\mu^2+\eta^2)^{-\frac{1}{2}}\,d\eta - i\pi H_0^{(1)}(a\mu)\right]d\rho\,\rho\,d\psi. \tag{A 1.1}$$

Now, recalling that μ is given by $\mu^2 = r^2+\rho^2-2r\rho\cos(\theta-\psi)$, if we set

$$r^2+\rho^2+\eta^2 = \alpha, \quad 2r\rho = \beta, \quad \text{and} \quad \psi-\theta = \gamma,$$

$$\int_0^{2\pi}\cos\psi\int_0^\infty e^{-a\eta}(\mu^2+\eta^2)^{-\frac{1}{2}}\,d\eta\,d\psi = \int_0^\infty e^{-a\eta}\int_0^{2\pi}\cos\psi\,[\alpha-\beta\cos(\psi-\theta)]^{-\frac{1}{2}}\,d\psi\,d\eta$$
$$= \int_0^\infty e^{-a\eta}\int_{-\theta}^{2\pi-\theta}\cos(\gamma+\theta)\,[\alpha-\beta\cos\gamma]^{-\frac{1}{2}}\,d\gamma\,d\eta$$
$$= \int_0^\infty e^{-a\eta}\left\{\cos\theta\int_0^{2\pi}\cos\gamma\,[\alpha-\beta\cos\gamma]^{-\frac{1}{2}}\,d\gamma - \sin\theta\int_0^{2\pi}\sin\gamma[\alpha-\beta\cos\gamma]^{-\frac{1}{2}}\,d\gamma\right\}d\eta$$
$$= 2\cos\theta\int_0^\infty e^{-a\eta}\int_0^\pi\cos\gamma[\alpha-\beta\cos\gamma]^{-\frac{1}{2}}\,d\gamma\,d\eta. \tag{A 1.2}$$

Here, replacing γ by $\pi-\delta$, we find

$$\int_0^\pi \cos\gamma[\alpha-\beta\cos\gamma]^{-\frac{1}{2}}\,d\gamma = -\int_\pi^0 \cos(\pi-\delta)\,[\alpha-\beta\cos(\pi-\delta)]^{-\frac{1}{2}}\,d\delta$$

$$= -\int_0^\pi \cos\delta[\alpha+\beta\cos\delta]^{-\frac{1}{2}}\,d\delta$$

$$= \frac{\alpha}{\beta}\int_0^\pi [\alpha+\beta\cos\delta]^{-\frac{1}{2}}\,d\delta - \frac{1}{\beta}\int_0^\pi [\alpha+\beta\cos\delta]^{\frac{1}{2}}\,d\delta.$$

From the identities in Byrd & Friedman (1954)

$$\int_0^\varphi [\alpha+\beta\cos\theta]^{-\frac{1}{2}}\,d\theta = 2(\alpha+\beta)^{-\frac{1}{2}}\,F(\tfrac{1}{2}\varphi,k)$$

and

$$\int_0^\varphi [\alpha+\beta\cos\theta]^{\frac{1}{2}}\,d\theta = 2(\alpha+\beta)^{\frac{1}{2}}\,E(\tfrac{1}{2}\varphi,k),$$

we have

$$\int_0^\pi \cos\gamma[a-\beta\cos\gamma]^{-\frac{1}{2}}\,d\gamma = 2\alpha\beta^{-1}(\alpha+\beta)^{-\frac{1}{2}}\,F(\tfrac{1}{2}\pi,k) - 2\beta^{-1}(\alpha+\beta)^{\frac{1}{2}}\,E(\tfrac{1}{2}\pi,k)$$

$$= 2\beta^{-1}(\alpha+\beta)^{-\frac{1}{2}}\,[\alpha K(k)-(\alpha+\beta)\,E(k)], \tag{A 1.3}$$

where k denotes the modulus of Jacobian elliptic functions, and $k^2 = 2\beta/(\alpha+\beta)$. In particular, when $\varphi = \frac{1}{2}\pi$, the integrals $F(\frac{1}{2}\pi,k) \equiv K(k)$ and $E(\frac{1}{2}\pi,k) \equiv E(k)$ are the complete elliptic integral of the first and second kind, respectively.

Further, from the identity (Byrd & Friedman 1954),

$$\int_0^\infty e^{-\rho t}J_1(rt)\,J_1(st)\,dt = (1/k\pi)\,(rs)^{-\frac{1}{2}}\,[(2-k^2)\,K(k)-2E(k)],$$

we find

$$\int_0^\infty e^{-\eta t}J_1(rt)\,J_1(\rho t)\,dt = \frac{1}{\pi}\left(\frac{\beta}{2}\,\frac{2\beta}{\alpha+\beta}\right)^{-\frac{1}{2}}\left[\left(2-\frac{2\beta}{\alpha+\beta}\right)K(k)-2E(k)\right]$$

$$= (2/\pi\beta)\,(\alpha+\beta)^{-\frac{1}{2}}\,[\alpha K(k)-(\alpha+\beta)\,E(k)].$$

It follows then from (A 1.3) that

$$\int_0^\pi \cos\gamma\,[\alpha-\beta\cos\gamma]^{-\frac{1}{2}}\,d\gamma = \pi\int_0^\infty e^{-\eta t}J_1(rt)\,J_1(\rho t)\,dt, \tag{A 1.4}$$

and the substitution of (A 1.4) in (A 1.2) yields

$$\int_0^{2\pi}\cos\psi\int_0^\infty e^{-a\eta}(\mu^2+\eta^2)^{-\frac{1}{2}}\,d\eta\,d\psi = 2\pi\cos\theta\int_0^\infty e^{-a\eta}\int_0^\infty e^{-\eta t}J_1(rt)\,J_1(\rho t)\,dt\,d\eta$$

$$= 2\pi\cos\theta\int_0^\infty J_1(rt)\,J_1(\rho t)\left[\int_0^\infty e^{-(a+t)\eta}\,d\eta\right]dt$$

$$= 2\pi\cos\theta\int_0^\infty \frac{J_1(rt)\,J_1(\rho t)}{a+t}\,dt. \tag{A 1.5}$$

Now, by (A 1.5), (A 1.3) can be expressed as

$$L[r, \theta, 0: f(r)\cos\theta] = -a\int_0^1 \rho f(\rho)\int_0^\infty \frac{J_1(rt)\,J_1(\rho t)}{a+t}\,dt\,d\rho\cos\theta$$

$$+\tfrac{1}{2}ia\int_0^1 \rho f(\rho)\int_0^{2\pi} H_0^{(1)}(a\mu)\cos\psi\,d\psi\,d\rho. \qquad \text{(A 1.6)}$$

It can be shown (Morse & Feshbach 1953) that $H_0^{(1)}(a\mu)$ has the expansions

$$H_0^{(1)}(a\mu) = J_0(ar)\,H_0^{(1)}(a\rho) + 2\sum_{n=1}^{\infty} J_n(ar)\,H_n^{(1)}(a\rho)\cos n(\theta-\psi), \quad \text{for} \quad \rho > r,$$

$$= J_0(a\rho)\,H_0^{(1)}(ar) + 2\sum_{n=1}^{\infty} J_n(a\rho)\,H_n^{(1)}(ar)\cos n(\theta-\psi), \quad \text{for} \quad \rho < r.$$

Hence, we find

$$\int_0^{2\pi} H^{(1)}(a\mu)\cos\psi\,d\psi = 0 + 2\pi\cos\theta\,J_1(ar)\,H_1^{(1)}(a\rho) + O(\cos 2\theta) \quad (\rho > r)$$

$$= 0 + 2\pi\cos\theta\,J_1(a\rho)\,H_1^{(1)}(ar) + O(\cos 2\theta) \quad (\rho < r). \qquad \text{(A 1.7)}$$

Finally, from (A 1.6) and (A 1.7), we obtain

$$L[r, \theta, 0: f(r)\cos\theta] = -a\int_0^1 \rho f(\rho)\left[\int_0^\infty \frac{J_1(rt)\,J_1(\rho t)}{a+t}\,dt\right]d\rho\cos\theta$$

$$+i\pi a\left[\int_0^r \rho f(\rho)\,J_1(a\rho)\,d\rho\,H_1^{(1)}(ar) + \int_r^1 \rho f(\rho)\,H_1^{(1)}(a\rho)\,d\rho\,J_1(ar)\right]\cos\theta$$

$$= l(f)\cos\theta. \qquad \text{(A 1.8)}$$

Appendix 2

For small values of the parameter a we develop the solution of the integral equation

$$f(r,\theta) + \frac{a}{2\pi}\int_0^{2\pi}\int_0^1 f(\rho,\psi)\,G(r,\rho,\theta,\psi\colon a)\,d\rho\,\rho\,d\psi = h^*(r,\theta), \qquad \text{(A 2.1)}$$

where

$$aG(r,\rho,\theta,\psi\colon a) = \sum_{n=1}^{\infty} P_n a^{n+1} + \sum_{n=1}^{\infty} Q_n a^{n+2}\log a. \qquad \text{(A 2.2)}$$

From inspection of (A 2.2) we shall consider the asymptotic expansion

$$f(r,\theta) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{ij}\,a^i(a\log a)^j, \qquad \text{(A 2.3)}$$

where the power products $a^i(a\log a)^j$ can always be ordered as to their rate of vanishing as the parameter a tends to zero, viz.

$$\lim_{a\to 0}\frac{a^{i'}(a\log a)^{j'}}{a^i(a\log a)^j} = 0 \qquad \text{(A 2.4)}$$

if $i'+j' > i+j$, or $i'+j' = i+j$ and $j > j'$. Now, the integral term in (A 2.2) can be written in power series as

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}[P_n f_{ij} a^{n+i+j+1}(\log a)^j + Q_n f_{ij} a^{n+i+j+2}(\log a)^{j+1}]\,d\rho\,\rho\,d\psi$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}[E_{lm} a^{l+m}(\log a)^m]\,d\rho\,\rho\,d\psi. \qquad \text{(A 2.5)}$$

Here indices are related as

$$n+i+j+1 = l+m \quad \text{and} \quad j = m,$$
$$n+i+j+2 = l+m \quad \text{and} \quad j+1 = m,$$

and therefore we find that in both cases $n = l-i-1$, and

$$E_{lm} = \sum_{i=0}^{l-1} P_{l-i-1} f_{im} + \sum_{i=0}^{l-1} Q_{l-i-1} f_{i,m-1}. \qquad \text{(A 2.6)}$$

Suppose the known function h^* has an expansion

$$h^*(r,\theta) = \sum_{l=0}^{\infty} h_l a^l,$$

then we may express (A 2.1) in the form

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} f_{lm} a^{l+m}(\log a)^m + \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} E_{lm} a^{l+m}(\log a)^m\,d\rho\,\rho\,d\psi = \sum_{l=0}^{\infty} h_l a^l. \qquad \text{(A 2.7)}$$

From (A 2.7) we find if $m = 0$

$$f_{l0} = h_l - \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \sum_{i=0}^{l-1} P_{l-i-1} f_{i0}\,d\rho\,\rho\,d\psi, \qquad \text{(A 2.8)}$$

and if $m \neq 0$

$$f_{lm} = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \sum_{i=0}^{l-1} (P_{l-i-1} f_{im} + Q_{l-i-1} f_{im-1})\,d\rho\,\rho\,d\psi. \qquad \text{(A 2.9)}$$

We observe that (A 2.8) and (A 2.9) are recurrence formulas by which the coefficients f_{ij} can be determined successively. For example, we obtain the first few terms from (A 2.8):

$$\left.\begin{aligned}
f_{00} &= h_0,\\
f_{10} &= h_1 - \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 (P_0 f_{00})\,d\rho\,\rho\,d\psi,\\
f_{20} &= h_2 - \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 (P_1 f_{00} + P_0 f_{10})\,d\rho\,\rho\,d\psi,\\
f_{30} &= h_3 - \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 (P_2 f_{00} P_1 f_{10} + P_0 f_{20})\,d\rho\,\rho\,d\psi,
\end{aligned}\right\} \qquad \text{(A 2.10)}$$

and, from (A 2.9), $\qquad f_m = 0 \quad$ for all m, $\qquad$ (A 2.11)

and

$$\left.\begin{aligned}
f_{11} &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 (P_0 f_{01} + Q_0 f_{00})\,d\rho\,\rho\,d\psi,\\
f_{12} &= 0,\\
f_{11} &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 (P_0 f_{11} + Q_1 f_{00} + Q_0 f_{10})\,d\rho\,\rho\,d\psi.
\end{aligned}\right\} \qquad \text{(A 2.12)}$$

This process ultimately leads to (A 2.3). We say $f(r,\theta\colon a)$ has an estimate of degree (i,j) if

$$f(r,\theta\colon a) = P(a, a\log a) + O[a^i(a\log a)^j], \tag{A 2.13}$$

where P is a polynomial of degree (i,j). From (A 2.2) we find

$$aG = -a^2\log a + O(a^2\log a);$$

hence, the product of aG with a polynomial of degree (i,j) is a polynomial of degree $(i+1, j+1)$ in addition to terms of $O[a^{i+1}(a\log a)^{j+1}]$.

Suppose that by computing $f_{00}, f_{10}, \ldots, f_{2m}$ we have shown $f(r,\theta\colon a)$ to have an estimate of degree (l,m). Substituting this estimate in the integral term in (A 2.7) we obtain a polynominal of degree $(l+1, m+1)$ plus terms of

$$O[a^{l+1}(a\log a)^{m+1}].$$

The right-hand side of (A 2.7) has the estimate of degree $(l, 0)$ for all l, hence (A 2.7) yields for $f(r,\theta\colon a)$ an estimate of degree $(l+1, m+1)$.

Now, retaining the terms with $h_0 = r\cos\theta$ in the coefficients f_{ij} we may write the asymptotic solution as

$$f(r,\theta\colon a) \sim f_{00} + af_{10} + a(a\log a)f_{11} + a^2 f_{20} + a^2(a\log a)f_{21} + a^3 f_{30}. \tag{A 2.14}$$

Appendix 3

Here we present another method for evaluating the damping coefficient of a pitching circular disk based on a consideration of the energy associated with radiating waves. Substitution of (C) of (2.12) into (4.5) yields

$$\frac{\bar{H}}{\rho^* \bar{a}^4 \sigma} = \operatorname{Im}\left\{\iint_{S^0} u u_y\, dS\right\}. \tag{A 3.1}$$

Note here that, since u_y is real, we have $uu_y = u\bar{u}_y = u\bar{u}_n$. The normalized damping coefficient is therefore given by

$$H = \frac{\bar{H}}{\rho \bar{a}^4 \sigma} = \operatorname{Im}\left\{\iint_{S^0} u u_n\, dS\right\}. \tag{A 3.2}$$

Let us consider a region D lying inside a vertical half cylinder which is bounded by a cylindrical wall C, the disk S^0, and a portion of the free surface S_f. Since the function u is harmonic in D, then in applying Green's theorem we observe that

$$\operatorname{Im}\left\{\iint_C u\bar{u}_n\, dS\right\} + \operatorname{Im}\left\{\iint_{S^0} u\bar{u}_n\, dS\right\} + \operatorname{Im}\left\{\iint_{S_f} u\bar{u}_n\, dS\right\} = 0. \tag{A 3.3}$$

From the free-surface condition (B^0) of (3.1), we have

$$\bar{u}_n = \bar{u}_y = a\bar{u}. \tag{A 3.4}$$

Hence $u\bar{u}_n = a|u|^2$ is real, and (A 3.3) reduces to

$$\operatorname{Im}\left\{\iint_{S^0} u\bar{u}_n\, dS\right\} = -\operatorname{Im}\left\{\iint_C u\bar{u}_n\, dS\right\}. \tag{A 3.5}$$

Thus, if r denotes the radius of C we can express (A 3.2) in terms of the energy of radiating waves as

$$H = -\operatorname{Im}\left\{\int_0^{2\pi}\int_{-\infty}^0 u\bar{u}_r\,dy\,r\,d\theta\right\}. \tag{A 3.6}$$

For a mode of pitch, the potential of radiating waves at large distances takes the form

$$u \sim F\cos\theta\, r^{-\frac{1}{2}}\, e^{ay+iar}. \tag{A 3.7}$$

Hence,

$$\bar{u}_r \sim -ia\bar{F}\cos\theta\, r^{-\frac{1}{2}}\, e^{ay-iar}. \tag{A 3.8}$$

Substituting (A 3.7) and (A 3.8) into (A 3.6), we find

$$H = a\int_0^{2\pi}\int_{-\infty}^0 |F|^2\cos^2\theta\, e^{2ay}\,dy\,d\theta = \tfrac{1}{2}\pi|F|^2. \tag{A 3.9}$$

On the other hand, from (3.3) we have

$$G \sim i\pi a\, e^{ay} H_0^{(1)}(a\mu) \quad \text{as} \quad r \to \infty. \tag{A 3.10}$$

Therefore the direct evaluation of the potential yields

$$\begin{aligned} u &\sim \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 f(\rho)\cos\psi\,[i\pi a\, e^{ay} H_0^{(1)}(a\mu)]\,d\rho\,\rho\,d\psi \\ &= ia\pi\, e^{ay} H_1^{(1)}(ar)\cos\theta\int_0^1 f(\rho)\,\rho J_1(a\rho)\,d\rho \\ &\approx (1-i)\,(a\pi)^{\frac{1}{2}}\, r^{-\frac{1}{2}}\cos\theta\, e^{ay+iar}\int_0^1 f(\rho)\,\rho J_1(a\rho)\,d\rho \quad (r > \rho), \end{aligned} \tag{A 3.11}$$

where $H_0^{(1)}(ar) \sim -(1+i)\,(a\pi r)^{-\frac{1}{2}}$ as $r \to \infty$.

Now, comparing (A 3.7) and (A 3.11) we find that

$$F = (1-i)\,(a\pi)^{\frac{1}{2}}\int_0^1 f(\rho)\,\rho J_1(a\rho)\,d\rho, \tag{A 3.12}$$

where f is a complex function.

Finally substituting (A 3.12) into (A 3.9) we obtain

$$\begin{aligned} H = {} & \tfrac{1}{2}\pi^2 a\left\{\int_0^1 [\operatorname{Re}\{f(\rho) + \operatorname{Im}(f(\rho)\}]\,\rho J_1(a\rho)\,d\rho\right\}^2 \\ & + \tfrac{1}{2}\pi^2 a\left\{\int_0^1 [\operatorname{Re}\{f(\rho)) - \operatorname{Im}(f(\rho)\}]\,\rho J_1(a\rho)\,d\rho\right\}^2, \end{aligned} \tag{A 3.13}$$

where $\operatorname{Re}[f(r)] = [f^0(r) + (\operatorname{Re}A)f^1(r)]$, and $\operatorname{Im}[f(r)] = [(\operatorname{Im}A)f^1(r)]$. With the solution (5.15) of the integral equation, the normalized damping coefficient can also be evaluated by (A 3.13).

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